

Today: ① Finish mins

② Transition to collective motion

①

Motility induced phase separation

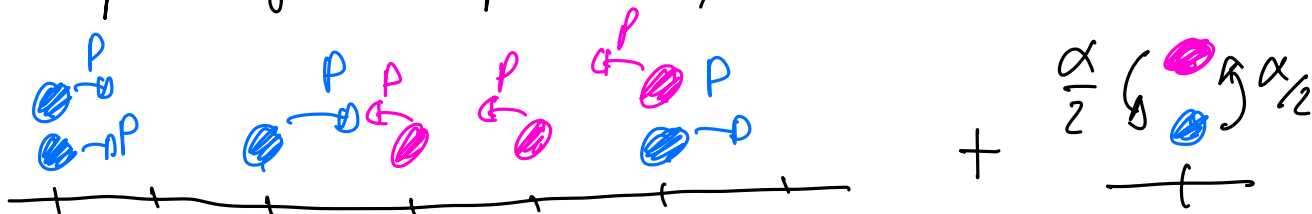
* Feedback between $P(\vec{r}) \propto \frac{1}{v(\vec{r})}$ & $v(\vec{r}) < 0 \Rightarrow$ phase separation.

* Hand waving criterion for phase separation $\partial_g (svG) < 0$

Can we show this on a real model?

7.7.2) RTPs on lattice in 1D

Here, we focus on the simpler case of α -lattice RTPs in 1D.
(inspired by [Thompson et al, 5. Stat-Mech 2011].)



Lattice gas model of non-interacting un- and-tumble bacteria.

Configurations $\varphi \Leftrightarrow$ occupancies (n_i^+, n_i^-) which describe the number of particles hopping to the right or to the left on site i .

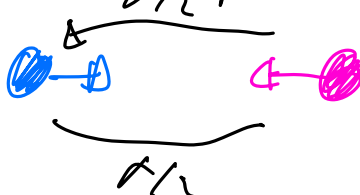
Add excluded volume interaction: partial exclusion

(2)

Hopping $i \rightarrow i+1$ for a right going particle $p(1 - \frac{n_{i+1}}{n_m})$
 $i \rightarrow i-1$ — left — $p(1 - \frac{n_{i-1}}{n_m})$

$n_i = n_i^+ + n_i^-$ is the total occupancy.

n_m is the maximal occupancy allowed on a lattice site

Tumbling rate 

Idea: Describe the dynamics of $\varphi_i^+ = \langle n_i^+ \rangle$ and show that homogeneous profiles are stable.

Master equation: $\partial_t P(\varphi) = \sum_{\varphi' \neq \varphi} W(\varphi' \rightarrow \varphi) P(\varphi') - W(\varphi \rightarrow \varphi') P(\varphi)$

Here $\varphi = \{n_i^+, n_i^-\} = \{n_1^+, n_1^-, n_2^+, n_2^-, \dots, n_L^+, n_L^-\}$

① φ' such that $W(\varphi' \rightarrow \varphi) \neq 0$

* $\varphi' = \{n_1^+, n_1^-, \dots, n_{i-1}^+, n_{i-1}^-, n_i^+ + 1, n_i^- - 1, \dots, n_L^+, n_L^-\} \equiv \{n_{i-1}^+ + 1, n_i^+ - 1\}$

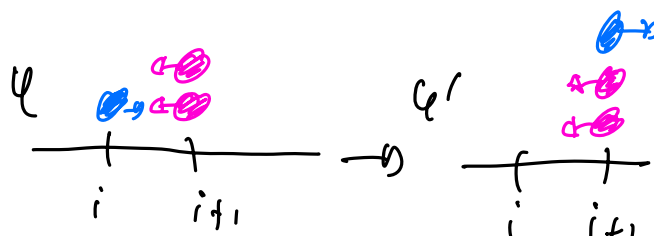
$\varphi' = n_{i-1}^+ + 1, n_i^+ - 1 \rightarrow \varphi$ at rate $W = p(n_{i-1}^+ + 1) (1 - \frac{n_i^-}{n_m})$
 and $n_j^+ = n_j^-$ same as in φ

* φ' as φ but with $n_{i+1}^- + 1, n_i^- - 1 \rightarrow \varphi$ at rate $W(\varphi' \rightarrow \varphi) = p(n_{i+1}^- + 1) (1 - \frac{n_i^+}{n_m})$

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* $\mathcal{Q}'$ as $\mathcal{Q}$ but with $\mu_i^+ + 1, \mu_i^- - 1$; then $W(\mathcal{Q}' \rightarrow \mathcal{Q}) = \frac{\alpha}{2} (\mu_i^+ + 1)$

* $\mathcal{Q}'$ as $\mathcal{Q}$ but with $\mu_i^- + 1, \mu_i^+ - 1$; then $W(\mathcal{Q}' \rightarrow \mathcal{Q}) = \frac{\alpha}{2} (\mu_i^- + 1)$

② $\mathcal{Q}'$ such that $W(\mathcal{Q} \rightarrow \mathcal{Q}') \neq 0$, e.g. 

* $\mathcal{Q}'$ as $\mathcal{Q}$ but with $\{\mu_i^+ - 1, \mu_{i+1}^+ + 1\}$, $W(\mathcal{Q} \rightarrow \mathcal{Q}') = \rho \mu_i^+ \left(1 - \frac{\mu_{i+1}}{\mu_\mu}\right)$

* --- $\{\mu_{i+1}^- - 1, \mu_i^- + 1\}$, $W(\mathcal{Q} \rightarrow \mathcal{Q}') = \rho \mu_{i+1}^- \left(1 - \frac{\mu_i}{\mu_\mu}\right)$

* --- $\{\mu_i^+ - 1, \mu_i^- + 1\}$, $W(\mathcal{Q} \rightarrow \mathcal{Q}') = \frac{\alpha}{2} \mu_i^+$

* --- $\{\mu_i^- - 1, \mu_i^+ + 1\}$, $W(\mathcal{Q} \rightarrow \mathcal{Q}') = \frac{\alpha}{2} \mu_i^-$

only indicating what changes

Master eq:

$$\begin{aligned} \partial_t P(\{\mu_i^+, \mu_i^-\}) = & \sum_i \rho (\mu_{i-1}^+ + 1) \left(1 - \frac{\mu_{i-1}}{\mu_\mu}\right) P(\{\mu_{i-1}^+ + 1, \mu_{i-1}^- - 1\}) \\ & + \rho (\mu_{i+1}^- + 1) \left(1 - \frac{\mu_{i+1}}{\mu_\mu}\right) P(\{\mu_{i+1}^- + 1, \mu_i^- - 1\}) \\ & + \frac{\alpha}{2} (\mu_i^+ + 1) P(\{\mu_i^+ + 1, \mu_i^- - 1\}) + \frac{\alpha}{2} (\mu_i^- + 1) P(\{\mu_i^- + 1, \mu_i^+ - 1\}) \\ & - \left[\rho \mu_i^+ \left(1 - \frac{\mu_{i+1}}{\mu_\mu}\right) + \rho \mu_i^- \left(1 - \frac{\mu_{i-1}}{\mu_\mu}\right) + \frac{\alpha}{2} \mu_i^+ + \frac{\alpha}{2} \mu_i^- \right] P(\{\mu_i^+, \mu_i^-\}) \end{aligned}$$

where $P(\{\mu_i^+ + 1, \mu_j^+ - 1\}) = P(\mathcal{Q}' = \mathcal{Q} \text{ with } \mu_i^+ \text{ replaced by } \mu_i^+ + 1 \text{ and } \mu_j^+ \text{ replaced by } \mu_j^+ - 1)$

From there, use $\langle n_i^{\pm} \rangle = \sum_{\{n_j\}} n_i^{\pm} P(\{n_j\})$ to get $\partial_t \langle n_i^{\pm} \rangle$ from the master equation $\Rightarrow$ good luck!

Evolution of observable: $\langle O(\varphi) \rangle = \sum_{\varphi} O(\varphi) P(\varphi, t) = \langle O \rangle_{\varphi}$

$$\partial_t \langle O \rangle_{\varphi} = \sum_{\varphi} O(\varphi) \partial_t P(\varphi, t) \quad \text{Master equation}$$

$$= \sum_{\varphi, \varphi'} O(\varphi) [W(\varphi' \rightarrow \varphi) P(\varphi') - W(\varphi \rightarrow \varphi') P(\varphi)]$$

$\varphi \leftrightarrow \varphi'$

$$= \sum_{\varphi, \varphi'} O(\varphi') W(\varphi \rightarrow \varphi') P(\varphi) - O(\varphi) W(\varphi \rightarrow \varphi') P(\varphi)$$

$$= \sum_{\varphi} \left[\sum_{\varphi'} (O(\varphi') - O(\varphi)) W(\varphi \rightarrow \varphi') \right] P(\varphi)$$

$$\partial_t \langle O \rangle_{\varphi} = \left\langle \underbrace{\sum_{\varphi'} (O(\varphi') - O(\varphi))}_{\Delta O(\varphi \rightarrow \varphi')} \underbrace{W(\varphi \rightarrow \varphi')}_{\text{rate at which this change occurs}} \right\rangle_{\varphi}$$

change in O from φ to φ'

Dynamics of the average occupancies: $\partial_t \langle n_i^{\pm} \rangle = ?$

Consider φ' that can be reached from $\varphi = \{n_i^{\pm}, n_j^{\pm}\}$
and compute $n_i^{\pm}(\varphi') - n_i^{\pm}(\varphi) = \Delta n_i^{\pm}$

Moves that impact $n_i^{\pm}$ are: . hops into or out of site i
 . trouble at site i

(5)

$$\begin{aligned} \partial_t \langle m_i^+ \rangle = & \langle \overbrace{(+1) \times p}^{g m_i^+} \overbrace{m_{i-1}^+ (1 - \frac{m_i}{m_m})}^w \rangle + \langle \overbrace{(-1) \times p}^{g m_i^+} \overbrace{m_i^+ (1 - \frac{m_{i+1}}{m_m})}^w \rangle \\ & + \langle \overbrace{(-1) \times \frac{\alpha}{2} m_i^+}^{g m_i^+} \rangle + \langle \overbrace{(+1) \times \frac{\alpha}{2} m_i^-}^{g m_i^+} \rangle \end{aligned}$$

$i-1 \rightarrow m_i$
tube $\rightarrow -$
 $i \rightarrow m_{i+1}$
tube $\rightarrow +$

$$\partial_t \langle m_i^+ \rangle = p \langle m_{i-1}^+ (1 - \frac{m_i}{m_m}) \rangle - p \langle m_i^+ (1 - \frac{m_{i+1}}{m_m}) \rangle - \frac{\alpha}{2} \langle m_i^+ \rangle + \frac{\alpha}{2} \langle m_i^- \rangle$$

$$\partial_t \langle m_i^- \rangle = p \langle m_{i+1}^- (1 - \frac{m_i}{m_m}) \rangle - p \langle m_i^- (1 - \frac{m_{i-1}}{m_m}) \rangle + \frac{\alpha}{2} \langle m_i^+ \rangle - \frac{\alpha}{2} \langle m_i^- \rangle$$

Comment: These equations involve $\langle m_{i-1}^+, m_i \rangle$, $\langle m_i^+, m_{i+1} \rangle$, etc $\Rightarrow$ not closed equations for $\langle m_i^+ \rangle$.

In practice N -points functions will involve $m+1$ point functions in their dynamics. We need approximations to get a solvable system.

Mean-field approximation: $\langle m_i^+ m_j^+ \rangle \simeq \langle m_i^+ \rangle \langle m_j^+ \rangle$

In general, this is quantitatively wrong, but often qualitatively right.

$$s_i^\pm = \langle m_i^\pm \rangle; \quad s_i = \langle m_i \rangle; \quad s_m = m_m$$

Dynamics: $\partial_t s_i^+ = p \left[s_{i-1}^+ \left(1 - \frac{s_i}{s_m} \right) - s_i^+ \left(1 - \frac{s_{i+1}}{s_m} \right) \right] - \frac{\alpha}{2} s_i^+ + \frac{\alpha}{2} s_i^- \quad (1)$

$$\partial_t s_i^- = p \left[s_{i+1}^- \left(1 - \frac{s_i}{s_m} \right) - s_i^- \left(1 - \frac{s_{i-1}}{s_m} \right) \right] + \frac{\alpha}{2} s_i^+ - \frac{\alpha}{2} s_i^- \quad (2)$$

Typical microscopic length $d_p = \frac{p}{\alpha} \Rightarrow$ expect variations of $s_i^\pm$ on this scale. If $\frac{p}{\alpha} \gg 1$, little differences between $s_i^\pm$ and $s_{i\pm 1}^\pm$.

$\Rightarrow$ Taylor expand $\phi_{i\pm 1} \simeq \phi(x) \pm \frac{1}{L} \nabla \phi(x) + \frac{1}{2L^2} \Delta \phi(x); \quad x = \frac{i}{L} \in [0,1]$

Cancelling out this expansion in (1) & (2) leads to $\nabla = \partial_x$; $\partial = \partial_x$ (6)

$$\begin{cases} \partial_\epsilon g^+(x) = -\frac{p}{L} (\partial_x g^+) \left(1 - \frac{g}{g_m}\right) + \frac{p}{L} g^+ \frac{\nabla g}{g_m} + \frac{p}{2L^2} \Delta g^+ \left(1 - \frac{g}{g_m}\right) + \frac{p g^+}{2L^2} \frac{\Delta g}{g_m} - \frac{\alpha}{2} g^+ + \frac{\alpha}{2} g^- & (3) \\ \partial_\epsilon g^-(x) = \frac{p}{L} (\partial_x g^-) \left(1 - \frac{g}{g_m}\right) - \frac{p}{L} g^- \frac{\partial_x g}{g_m} + \frac{p}{2L^2} (\partial_{xx} g^-) \left(1 - \frac{g}{g_m}\right) + \frac{p}{2L^2} g^- \frac{\partial_{xx} g}{g_m} + \frac{\alpha}{2} g^+ - \frac{\alpha}{2} g^- & (4) \end{cases}$$

$$g(x) = g^+(x) + g^-(x); m(x) = g^+(x) - g^-(x)$$

$$(3) + (4) \Rightarrow \partial_\epsilon g(x) = -\partial_x \left[\frac{p}{L} m \left(1 - \frac{g}{g_m}\right) - \frac{p}{2L^2} \partial_x g \right] \quad (5)$$

$$(3) - (4) \Rightarrow \partial_\epsilon m(x) = -\alpha m(x) - \partial_x \left[\frac{p}{L} g \left(1 - \frac{g}{g_m}\right) \right] + \frac{p}{2L^2} (\partial_{xx} m) \left(1 - \frac{g}{g_m}\right) + \frac{p}{2L^2} m \left(\frac{\partial_{xx} g}{g_m} \right) \quad (6)$$

Analysis of (5) - (6) in the large L limit:

$L \rightarrow \infty$ (5): $\partial_\epsilon g = 0 \Rightarrow$ Nothing happens in a time of order 1 for a conserved field in the large size limit.

Speed up time $t = L\tau$

$$(5) \Rightarrow \frac{dg}{d\tau} = -\partial_x \left[p m(x) \left(1 - \frac{g(x)}{g_m}\right) - \frac{p}{2L} \partial_x g \right] \quad (5)'$$

$$(6) \quad m(x) = -\alpha L m - p \partial_x \left[g \left(1 - \frac{g}{g_m}\right) \right] + \frac{p}{2L} (\partial_{xx} m) \left(1 - \frac{g}{g_m}\right) + \frac{p}{2L} m \partial_{xx} g$$

If $\partial_{xx} g$ & $\partial_{xx} m$ are finite, then the last two terms are negligible.

$$m(x) = -\alpha L m(x) - p \partial_x \left[g \left(1 - \frac{g}{g_m}\right) \right]$$

exponential relaxation
in time scale

$$\tau \sim \frac{1}{\alpha L}$$

external source

$\Rightarrow$

$$m(x) \approx -\frac{p}{\alpha L} \partial_x \left[g \left(1 - \frac{g}{g_m}\right) \right] \quad \text{for } \tau \sim O(1)$$

Injecting $m(x)$ back into (5)' then leads to

$$\dot{s} = \partial_x \left[\frac{p^2}{\alpha L} \left(1 - \frac{sg(x)}{s_m} \right) \partial_x \left[s \left(1 - \frac{s}{s_m} \right) \right] + \frac{p}{2L} \partial_x s \right] \quad (7)$$

Again $\dot{s} \approx 0$ for $\tau \sim O(1) \Rightarrow \tau = \tilde{\tau}$

$$\begin{aligned} \frac{dsg(x)}{d\tilde{\tau}} &= \partial_x \left[\left(\frac{p}{2} + \frac{p^2}{\alpha} \left(1 - \frac{s}{s_m} \right) \left(1 - \frac{2s}{s_m} \right) \right) \partial_x s \right] \\ &\equiv \partial_x \left[D_{eff}(sg(x)) \partial_x (s) \right] \end{aligned}$$

At the macroscopic $x = \frac{\tilde{x}}{L}$; $\tilde{\tau} = \frac{t}{L^2}$; we obtain an effective dynamics for the density field which is a non-linear diffusion equation.

If $D_{eff}(s_0) > 0 \Rightarrow$ small fluctuations around s_0 relax to 0.

— $D_{eff}(s_0) < 0 \Rightarrow$ ————— get amplified

$$(\dot{s} = D \Delta s \xrightarrow{\epsilon \rightarrow 0} \dot{s} = (-D) \Delta s)$$

$\Rightarrow$ linear instability

$$D_{eff} < 0 \Leftrightarrow \left(\frac{2s}{s_m} - 1 \right) \left(1 - \frac{s}{s_m} \right) \geq \frac{\alpha}{2p} = \frac{1}{2l_p} \Rightarrow \text{persistence length}$$

$$l_p \gg 1 \Rightarrow \text{instability for } s \geq \frac{s_m}{2}$$

Comment: hopping rate $\Leftrightarrow v(s) = p \left(1 - \frac{s}{s_m} \right)$ } the result of our computations is consistent with the hard-waving argument.

$$\frac{v'}{v} \leq -\frac{1}{s} \Leftrightarrow s \geq \frac{s_m}{2}$$

What remains is to characterize the phase coexistence emerging from this instability $\Rightarrow$ Solon et al., NJP 2018. 8